International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Orthosymmetric Ortholattices and Rickart *-Rings

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Received October 14, 1994

Orthosymmetric ortholattices (OSOLs) have been introduced in order to approximate ortholattices of closed subspaces of a Hilbert space. In this paper, some new properties of OSOLs are proved and the main result states that lattices of projections of Rickart *-rings, satisfying $2x = 0$ implies $x = 0$, carry a natural structure of OSOL.

INTRODUCTION AND MOTIVATION

A purpose of quantum logic is the characterization of the variety generated by Hilbert lattices $\mathscr{C}(H)$, which are lattices of all closed subspaces of a Hilbert space H. A motivation is the research for a set of axioms for a syntactical presentation of quantum logic since Kalmbach (1983, §4, exercise 21) proved that this variety (in the language of ortholattices) is strictly contained in the class of all OMLs (orthomodular lattices). Kalmbach used the ortho-Arguesian law, an equation discovered by A. Day, which fails in some OMLs and is valid in all Hilbert lattices. At the same time, R. Godowski, R. Greechie, and R. Mayet obtained, by means of methods using states, an abundance of varieties of OMLs containing the one generated by Hilbert lattices. More recently, R. Mayet has suggested a new idea: Hilbert lattices have a very rich structure and, besides the classical operations of meet, join, and orthocomplementation, they possess other natural operations and so they can be investigated in languages extending the language of ortholattices. For example, there exists an operation which associates to the closed subspaces X and Y the closed subspace symmetrical to Y with respect to X. Mayet (1991) gives a list of axioms verified by this operation; OMLs satisfying these axioms are called OSOLs (orthosymmetric ortholattices). The interest of the axioms of R. Mayet has been confirmed in Hamhalter and Navara

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0020-7748/95/0800-1289$07.50/0

(1991): if the dimension of the underlying Hilbert space H is greater than 3, there exists a unique structure of OSOL on $\mathscr{C}(H)$, the natural one defined by orthogonal symmetries. The power of the method is also illustrated by Mayet and Pulmannová (1994), where a generalization of the structure of OSOL allows one to distinguish complex Hilbert lattices and real or quaternionic ones by means of equations.

The purpose of Section 1 of this paper is to prove some new properties related to the lattice structure of OSOLs. Some of them are already known in the particular case of projection lattices of associative or Jordan operator algebras and this fact points out another interest of OSOLs: to give a common algebraic setting for the study of symmetries in operator algebras. Section 2 contains the main result, which states that projection lattices of Rickart *-rings, satisfying the condition $2x = 0$ implies $x = 0$, carry a natural structure of OSOL. The last section is devoted to examples and open questions.

For notions concerning orthomodular lattices, see Kalmbach (1983). For rings with involution, Berberian (1972) is a standard reference. In an OML, we denote by ϕ_a the Sasaki projection on a defined by $\phi_a(b) = a \wedge (a^\perp \vee b)$ and aCb means that a and b commute.

1. GENERAL PROPERTIES OF OSOLS

By definition (Mayet, 1991; Mayet and Pulmannová, 1994) an OSOL is an ortholattice L equipped with a binary operation S satisfying the following axioms, where $S(a, b)$ is denoted by $S_a(b)$:

- $OSOL_1$: Every mapping S_a is an involuntary automorphism of the ortholattice L and $S_a \circ S_b \circ S_a = S_{S_a(b)}$.
- $OSOL_2$: $a \perp b$ implies $S_a \circ S_b = S_b \circ S_a = S_{a \vee b}$.
- $OSOL_3$: $S_a(b) = b$ if and only if aCb.

Every OSOL is an OML and notice that $OSOL_3$ replaces the original axiom of Mayet (1991): $a \vee \phi_b(a) = a \vee S_b(a)$. The equivalence of these two axioms is proved in Mayet and Pulmannová (1994).

Two elements a and b of an OSOL L are said to be *exchanged by a symmetry* if there exists $c \in L$ such that $b = S_c(a)$. The following result has already been proved in the particular framework of operator algebras; see, for example, Topping (1965), Theorem 7.

Proposition 1. Two elements of an OSOL L exchanged by a symmetry are strongly perspective.

Proof. Let a and b two elements of L. A direct computation, using some results of Mayets (1991), leads to

$$b \vee \phi_a(b) = S_a(b) \vee \phi_a(b) = b \vee S_a(b)$$

$$b \wedge \phi_a(b) = S_a(b) \wedge \phi_a(b) = b \wedge a$$

and thus a common complement of b and $S_a(b)$ in $[0, b \vee S_a(b)]$ is

$$c = \phi_a(b) \wedge (a \wedge b)^{\perp} = a \wedge (a^{\perp} \vee b) \wedge (a^{\perp} \vee b^{\perp})$$

In the next proposition, we give formulas for the central cover, denoted by $|a|$, of an element a of an OSOL which is a complete lattice. Similar properties are known in operator algebras.

Proposition 2. In a complete OSOL L we have

$$\bigvee_{x \in L} S_x(a) = \bigvee_{x \in L} \phi_x(a) \qquad \text{and} \qquad |a| = \bigvee_{x_i \in L} S_{x_1} \cdots S_{x_n}(a)$$

If, moreover, L has the relative center property, then

$$|a| = \bigvee_{x \in L} S_x(a)$$

Proof. The first identity is a straightforward consequence of $S_a(a) = a$, $\phi_a(a) = a$, and $a \vee S_x(a) = a \vee \phi_x(a)$. Now let

$$h = \bigvee_{x_i \in L} S_{x_1} \cdots S_{x_n}(a)$$

For all $y \in L$, we have

$$S_y S_{x_1} \cdots S_{x_n}(a) \leq h$$

and thus $S_y(h) \leq h$, which implies yCh. The element h is central and, by using

$$S_{x_1} \cdots S_{x_n}(a) \leq S_{x_1} \cdots S_{x_n}(|a|) = |a|$$

we have $h \leq |a|$ and, as $S_a(a) = a$, we conclude $h = |a|$. The last formula is a consequence of Chevalier (1991), Proposition 10.

2. OSOL STRUCTURES IN PROJECTION LATTICES OF RICKART *-RINGS

In Hamhalter and Navara (1991) and Mayet (1991) the structure of OSOL in Hilbert lattices is described in the setting of subspaces and the authors consider for each closed subspace X of a Hilbert lattice H the mapping σ_X which associates to a closed subspace Y the subspace $\sigma_X(Y)$, symmetrical to Y with respect to X. The OML $C(H)$ is isomorphic to the lattice Proj(H) of all orthogonal projections of H; an isomorphism is the mapping which associates to a closed subspace X the projection admitting X as range. If p

and q are two projections of H, with respective ranges X and Y, then the range of the projection $(2p - 1)q(2p - 1)$ is $\sigma_X(Y)$. Thus in the setting of projections, the mapping S_p defined by $S_p(q) = (2p - 1)q(2p - 1)$ replaces σ_X. More precisely, we have the following result.

Proposition 3. For every idempotent p of a ring A, let S_p be the mapping defined for all $x \in A$ by

$$S_p(x) = (2p - 1)x(2p - 1)$$

Then:

1. S_p is an involuntary automorphism for the ring structure of A.
2. S_p conserves the set Idem(A) of all idempotents of A and is an order automorphism of Idem(A) ordered by $p \leq q$ if and only if $pq = qp = p$.
3. If, moreover, A is a ring with an involution $x \mapsto x^*$ and if p is a projection, then S_p conserves the set Proj(A) of all projections of A and is an involutary automorphism for the structure of the weak generalized orthomodular poset (WGMOP) of Proj(A).

The proof is routine and information about WGMOP may be found in Mayet-Ippolito (1991). The next proposition shows that the ring A, equipped with the set of involutary automorphisms S_p, $p \in$ Idem(A), satisfies properties very close to these of an OSOL.

Proposition 4. With notation as in Proposition 3 and if p and q are two idempotents of A, then:

1. $S_p \circ S_q \circ S_p = S_{S_p(q)}$.
2. $pq = qp = 0$ implies $S_p \circ S_q = S_q \circ S_p = S_{p+q}$.
3. $pq = qp$ implies $S_p(q) = q$ and $S_p(q) = q$ implies $2pq = 2qp$.

Proof. 1. For $x \in A$, we have

$$\begin{aligned} S_p \circ S_q \circ S_p(x) &= (2p - 1)(2q - 1)(2p - 1)x(2p - 1)(2q - 1)(2p - 1) \\ &= S_p(2q - 1)xS_p(2q - 1) \\ &= [2S_p(q) - 1]x[2S_p(q) - 1] = S_{S_p(q)}(x) \end{aligned}$$

2. If $pq = qp = 0$, then $p + q$ is an idempotent and $(2p - 1)(2q - 1) = -[2(p + q) - 1]$. Thus

$$\begin{aligned} S_p \circ S_q(x) &= (2p - 1)(2q - 1)x(2q - 1)(2p - 1) \\ &= [2(p + q) - 1]x[2(p + q) - 1] = S_{p+q}(x) \end{aligned}$$

3. Clearly, if $pq = qp$, then $S_p(q) = q$. Conversely, $S_p(q) = q$ implies $4pqp = 2pq + 2qp$ and right and left multiplication by the idempotent p yields $4pqp = 2pqp + 2qp$ and $4pqp = 2pq + 2pqp$. Therefore, $2pq = 2qp$.

Notice that part 3 of the previous proposition contains a difficulty: $S_p(q) = q$ only implies $2pq = 2qp$. For the more convenient conclusion $pq = qp$, it will be necessary to assume something about A, for example, $2x = 0$ implies $x = 0$.

Recall that a Rickart *-ring is a ring with involution in which the right annihilator of every element is a principal right ideal generated by a projection (Berberian, 1972).

Theorem 1. Every projection lattice of a Rickart *-ring A, satisfying the condition $2x = 0$ implies $x = 0$, has a natural structure of an OSOL.

Proof. Consider the family of involutary automorphisms of Proj(A) $\{S_p \mid p \in \text{Proj}(A)\}$. It is easy to see that Proj(A) is an OSOL, by using Propositions 3 and 4 and the two following results fulfilled by every Rickart *-ring:

- Two projections p and q are orthogonal if and only if $pq = qp = 0$ and, for two orthogonal projections, $p \vee q = p + q$.
- Two projections p and q commute (in the sense of OML theory) if and only if $pq = qp$.

3. REMARKS, EXAMPLES, AND QUESTIONS

1. Lattices of projection of von Neumann algebras, Rickart C^*-algebras (C^*-algebras which are Rickart *-rings), and AW^*-algebras (Rickart C^*-algebras with a complete lattice of projections) are OSOLs. In case of von Neumann algebras a different proof may be found in Mayet (1991).

2. There exists a Jordan version of Theorem 1: every lattice of idempotents of a JBW-algebra is an OSOL. A sketch of a proof is as follows. Consider, for each idempotent p of a JBW-algebra A [see Hanche-Olsen and Størmer (1984) for information] with a Jordan product denoted by $\circ$, the mapping S_p defined by $S_p = U_{2p-1}$, where $U_a(x) = 2a \circ (a \circ x) - a^2 \circ x$. Mappings S_p are involutary automorphisms of the Jordan algebra A and they conserve the OML Idem(A) of all idempotents of A. Notice that in a special Jordan algebra, where the associative product is denoted by juxtaposition, $S_p(x) = (2p - 1)x(2p - 1)$ and OSOL_1, OSOL_2, OSOL_3 involve only two or three variables. Therefore, the Shirshov–Cohn and Macdonald theorems may be used to reduce the Jordan case to the associative one. The characterization of commutativity in Idem(A) given in Chevalier (1994), Proposition 13, is also useful.

3. What happens if the rings A does not satisfy $2x = 0$ implies $x = 0$? Notice that a Boolean algebra A is a Rickart *-ring satisfying $2x = 0$ for all

x and there exist a unique OSOL structure on A = Proj(A) defined by $S_p(q) = q$ (Mayet, 1991).

4. Hilbert lattices possess a uniquely determined OSOL structure if the dimension of the underlying Hilbert space is greater than 3. Does this result generalize to some lattices of projections of Rickart *-rings? Notice that if L is the projection lattice of the Rickart *-ring of all endomorphisms of the linear space R^2, then there is more than one structure of OSOL on L. Define, for any atom $a \in L$, $S_a(b) = b^{\perp}$ if b is an atom such that $b \neq a$, $b \neq a^{\perp}$ and $S_a(b) = b$ otherwise (Mayet, 1991). We obtain in this way a structure of OSOL on a projection lattice of a Rickart *-ring different from the natural one.

5. Kaplansky introduced in a Rickart *-ring the square-root axiom, namely:

(SR) For each element x, there exists r in the double commutant of $\{xx^*\}$ such that $r^* = r$ and $xx^* = r^2$.

Every C^*-algebra satisfies the SR axiom and, in the presence of the SR axiom, there exist many results related to the exchange by symmetry (Maeda, 1975; Maeda and Holland, 1976). What are the nonclassical Hilbert spaces satisfying this axiom?

ACKNOWLEDGMENT

I would like to thank Sylvia Pulmannová for a fruitful discussion related to the axioms of OSOLs.

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